

# GENERALIZED QUON STATISTICS

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## Abstract

Generalized quons interpolating between Bose, Fermi, para-Bose, para-Fermi, and anyonic statistics are proposed. They follow from the R-matrix approach to deformed associative algebras. It is proved that generalized quons have the same main properties as quons. A new result for the number operator is presented and some physical features of generalized quons are discussed in the limit  $|q_{ij}^2| \rightarrow 1$ .

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Quons were proposed by Greenberg [1,2] as particles interpolating between bosons and fermions. Quonic intermediate statistics is an example of infinite statistics in which any representation of the symmetric group can occur. It was pointed out that quons offered a possibility for a small violation of the Pauli exclusion principle, at least in nonrelativistic theory [2,3]. The quon algebra, interpolating between bosonic and fermionic oscillators, was postulated [2] as

$$a_i a_j^\dagger - q a_j^\dagger a_i = \delta_{ij}, \quad q \in R. \quad (1)$$

Here  $i, j$  are discrete indices and the parameter  $q$  is a real number. The main properties of quons are as follows:

- (i) Norms are positive definite for  $-1 \leq q \leq 1$ .
- (ii) For  $q^2 \neq 1$ , the commutation relations do not exist between annihilation (creation) operators  $a_i, a_j$  ( $a_i^\dagger, a_j^\dagger$ ), i.e., there are  $n!$  linearly independent states  $a_{i_1} \dots a_{i_n} |0\rangle$  for different permutations of fixed indices  $1, 2, \dots, n$ .
- (iii) The number operator exist in the form of an infinite series expanded in powers of creation and annihilation operators, with complicated coefficients diverging when  $q^2 \rightarrow 1$ .
- (iv) The theory is nonlocal, but the TCP theorem and the clustering property hold in relativistic quon theories.

In a recent Letter [4] Mishra and Rajasekaran proposed a  $q$ -deformed algebra of creation and annihilation operators with ordered indices in which the deformation parameter was complex. The  $q$ -deformed algebra was defined by the following equations:

$$a_i a_j^\dagger - q a_j^\dagger a_i = 0 \text{ for } i < j$$

$$a_i a_i^\dagger - p a_i^\dagger a_i = 1,$$

where  $q$  and  $p$  are complex and real parameters, respectively. Consequences of this complex  $q$ -mutator algebra were studied and its relation to "fractional" statistics was pointed out.

Anyons were proposed [5,6] as particles in  $2 + 1$  dimension that also interpolate between bosons and fermions. It was shown [7] that multivalued anyons can be treated as a kind of quons with the multivalued unimodular parameter  $q = e^{i\lambda\Delta}$ ,  $\lambda$  being a real statistical parameter and  $\Delta = \pi + 2\pi z$ ,  $z \in \mathbb{Z}$ .

In this paper we propose generalized quons as particles interpolating between all kinds of statistics simultaneously. For a special choice of  $q$  parameters, one finds any particular statistics. Generalized quons in principle allow for a small violation of any statistics. We point out that, generally, the  $q$  parameter can depend on the pair of indices  $i, j$  in the product  $a_i a_j^\dagger$  in Eq.(1). Moreover,  $q_{ij}$  can be complex numbers, with  $q_{ij}^* = q_{ji}$ . Hence the global parameter  $q$  becomes a Hermitian matrix. We point out that generalized quons follow from the R-matrix approach to deformed algebras. We show that all properties of quons hold for generalized quons. A new result for the number operator is obtained. Some physical features of generalized quons are discussed in the limit  $|q_{ij}|^2 \rightarrow 1$ .

We propose the generalized quon algebra as

$$a_i a_j^\dagger - q_{ij} a_j^\dagger a_i = \delta_{ij}, \quad q_{ij}^* = q_{ji}, \quad i, j \in S, \quad (2)$$

where  $i, j$  denote sites of the discrete totally ordered lattice  $S$  in arbitrary dimen-

sion. This algebra interpolates between bosons (for  $q = 1$ ), fermions (for  $q = -1$ ), multivalued anyons [7] (for  $q = e^{i\lambda\pi(1+2z)}$ ,  $z \in Z$ ), single-valued anyons [8,9] (for  $q(\vec{r} - \vec{r}') = \pm e^{i\lambda[\theta_0(\vec{r}-\vec{r}') - \theta_0(\vec{r}'-\vec{r})]}$ , where  $\theta_0(\vec{r})$  is the polar angle and  $+$ ( $-$ ) corresponds to the Bose (Fermi)-type transformation). If  $q_{ij} = q$  for  $i < j$ , where  $q$  is a complex parameter, and  $q_{ii} = p$ , where  $p$  is a real parameter, we obtain the  $q$ -deformed algebra of Ref.[4].

In order to show that para-Bose and para-Fermi statistics [10] are also contained in the general quon algebra, Eq.(2), let us consider a set of operators  $a_i^\alpha$ ,  $\alpha = 1, \dots, p$ , satisfying the algebra

$$a_i^\alpha a_j^{\beta\dagger} - q_{\alpha\beta} a_j^{\beta\dagger} a_i^\alpha = \delta_{\alpha\beta} \delta_{ij}, \quad q_{\alpha\beta}^* = q_{\beta\alpha}. \quad (3)$$

Then  $q_{\alpha\beta} = \pm 1$  for  $\alpha = \beta$  and  $q_{\alpha\beta} = \mp 1$  for  $\alpha \neq \beta$  correspond to Green's para-Bose (para -Fermi) type of oscillators for the upper (lower) sign.

We point out that the generalized-quon algebra, Eq.(2), is just a special type of associative algebra; namely, the general commutation algebra of oscillators can be written in the R-matrix approach [11,12]:

$$\begin{aligned} a_i a_j - R_{ij,kl} a_l a_k &= 0, \\ a_i a_j^\dagger - R'_{ki,jl} a_k^\dagger a_l &= Q_{ij}. \end{aligned} \quad (4)$$

Here  $R$  and  $R'$  are matrices with complex entries, whereas  $(Q_{ij})$  is a set of operators with  $Q_{ij}^\dagger = Q_{ji}$ . In the following we restrict ourselves to the case  $Q_{ij} = \delta_{ij}$ .

In order that the algebra in Eq.(4) be associative, the following condition have to be satisfied:

(a) the Yang-Baxter equation (summation over repeated indices is assumed)

$$R_{jk,xy}R_{iy,zb}R_{zx,am} = R_{ij,xy}R_{xk,az}R_{yz,mb}, \quad (5)$$

(b)

$$R_{jk,xy}R'_{iy,zb}R'_{zx,am} = R'_{ij,xy}R'_{xk,az}R_{yz,mb}, \quad (6)$$

(c) the Hecke condition, symbolically written as

$$(\hat{R} - 1)(\hat{R}' + 1) = 0, \quad (7)$$

where  $\hat{R} = PR$ ,  $\hat{R}' = PR'$  and  $P$  is the permutation operator  $P_{ij,kl} = \delta_{il}\delta_{jk}$ ,

(d) hermiticity, i.e., that  $a_i^\dagger$  is the hermitian conjugate of  $a_i$  (and vice versa):

$$R'_{ij,kl} = R_{lk,ji}^*, \text{ i.e., } (\hat{R}')^\dagger = \hat{R}'. \quad (8)$$

Solutions of Eq.(5-8), corresponding to Bose, para-Bose, Fermi, para-Fermi, and anyonic statistics are

$R = R' = \pm 1$ , upper (lower) sign for bosons (fermions),

$R = R' = e^{i\lambda\pi(1+2z)} \cdot 1$ ,  $z \in Z$ , for multivalued anyons [7],

$$\begin{aligned} R(\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4) &= R'(\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4) \\ &= \pm e^{i\lambda[\theta_0(\vec{r}_1 - \vec{r}_2) - \theta_0(\vec{r}_2 - \vec{r}_1)]} \delta(\vec{r}_1 - \vec{r}_3) \delta(\vec{r}_2 - \vec{r}_4) \end{aligned}$$

for single-valued anyons [8,9] with  $+$ ( $-$ ) for the Bose (Fermi)-type transformation,

and

$$R = R', \quad R_{\alpha i, \beta j, \gamma k, \delta l} = q_{\alpha\beta} \delta_{\alpha\gamma} \delta_{\beta\delta} \delta_{ik} \delta_{jl},$$

where  $\alpha, \beta, \gamma, \delta = 1, \dots, p$  and  $q_{\alpha\beta} = \pm(2\delta_{\alpha\beta} - 1)$ ,  $+$ ( $-$ ) corresponding to Green's

oscillators of the para-Bose (para-Fermi) type [10].

However, there is also another solution of Eqs.(5-8),  $R = P$ , i.e.,  $\hat{R} = 1$ , and an arbitrary Hermitian matrix  $R'$ . Specially, we put

$$(R')_{ki,jl} = q_{ij}\delta_{jk}\delta_{il}, \quad q_{ij} = q_{ji}^*. \quad (9)$$

The corresponding associative algebra is just the generalized-quon algebra in Eq.(2). There are no relation between  $a_i, a_j$  (or  $a_i^\dagger, a_j^\dagger$ ) operators. When  $R'$  approaches one of the above special solutions  $R = R'$ , quons approach bosons, fermions, anyons, etc., allowing for a small violation of the corresponding approaching statistics. In the exact limit, the relations between  $a_i, a_j$  (or  $a_i^\dagger, a_j^\dagger$ ) appear.

Finally, we have proved that all the main properties of quons still hold for generalized quons.

(i) The norms are positive definite for  $|q_{ij}| < 1$ .

We assume the existence of a vacuum state  $|0\rangle$  and its dual  $\langle 0|$  satisfying

$$a_i|0\rangle = 0, \quad \langle 0|a_i^\dagger = 0, \quad \forall i \in S. \quad (10)$$

The Fock-like space is constructed in the usual way. The one-particle states are  $a_i^\dagger|0\rangle$ ,  $i \in S$ , and, generally, the n-particle states are of the form  $a_{i_1}^\dagger \cdots a_{i_n}^\dagger|0\rangle$ ,  $i_1, \dots, i_n \in S$ . When the indices  $i_1, \dots, i_n$  are mutually different there may exist maximally  $n!$  linearly independent states specified by permutations of  $i_1, \dots, i_n$ . If  $n_1, \dots, n_a$  are multiplicities of equal indices appearing in an ordered sequence

$i_1, \dots, i_n$  ( $i_1 \leq i_2 \leq \dots \leq i_n$ ) satisfying  $\sum_{\alpha=1}^a n_\alpha = n$ , then there may exist maximally  $\frac{n!}{n_1! \dots n_a!}$  linearly independent states specified by permutations of  $i_1, \dots, i_n$ . The dual (bra) states are defined by  $\langle 0 | a_{i_n} \dots a_{i_1}, i_1, \dots, i_n \in S$ .

Let us define the matrix  $A$  of inner products with the matrix elements :

$$A_{i_1, \dots, i_m; j_1, \dots, j_n} = \langle 0 | a_{i_m} \dots a_{i_1} a_{j_1}^\dagger \dots a_{j_n}^\dagger | 0 \rangle. \quad (11)$$

This is in fact the vacuum matrix element of any polynomial  $a_{i_m} \dots a_{i_1} a_{j_1}^\dagger \dots a_{j_n}^\dagger$ , that can be calculated using Eqs(2),(10). The matrix  $A$  is hermitian and block-diagonal. If the lattice  $S$  is finite with number of sites  $D$ , then there are

$$\binom{D+n-1}{n}$$

different blocks (of size  $\leq n!$ ) in the  $n$ -particle sector. A generic block  $A^{(i_1 \dots i_n)}$  is characterized by mutually different ordered indices  $i_1, \dots, i_n \in S$  ( $i_1 < i_2 < \dots < i_n$ ),  $n \leq D$ , from which all other blocks in the  $n$ -particle sector can be obtained using a suitable specification. The  $A^{(i_1, \dots, i_n)}$  matrix is an  $n! \cdot n!$  matrix, whose diagonal matrix elements are equal to 1. The arbitrary matrix element  $(\pi, \sigma)$ , i.e.,  $i_{\pi(1)}, \dots, i_{\pi(n)}; i_{\sigma(1)} \dots i_{\sigma(n)}$ , where  $\pi$  and  $\sigma$  are permutations acting on positions  $1, 2 \dots n$  ( $\pi$  denotes the row and  $\sigma$  the column of the matrix  $A^{(i_1, \dots, i_n)}$ ) is given by

$$A_{\pi, \sigma}^{(i_1, \dots, i_n)} = \prod_{\alpha, \beta} q_{i_\alpha i_\beta}. \quad (12)$$

Here the product is over all pairs  $\alpha, \beta = 1, \dots, n$  satisfying  $\pi^{-1}(\alpha) < \pi^{-1}(\beta)$  and  $\sigma^{-1}(\alpha) > \sigma^{-1}(\beta)$ . For  $q_{ij} = q \in \mathcal{R}$ ,  $\forall i, j \in S$ , Eq.(12) reproduces the result

$A_{\pi,\sigma}^{(i_1,\dots,i_n)} = q^{I(\sigma^{-1}\cdot\pi)}$ , where  $I$  denotes the number of inversions in permutation  $\sigma^{-1}\cdot\pi$ , [2,13].

For example, for  $n = 2$ , Eq.(12) gives

$$A^{(i_1,i_2)} = \begin{pmatrix} 1 & q_{i_1 i_2} \\ q_{i_2 i_1} & 1 \end{pmatrix}, i_1, i_2 \in S, i_1 \neq i_2. \quad (13)$$

Note that if  $i_1 = i_2 = i$  the block matrix reduces to the one-by-one matrix  $A^{(i,i)} = 1 + q_{ii}$ ,  $i \in S$ .

Now we analyze the positivity for the norm (of all vectors) in the Fock-like space.

The norm of any one-particle state is positive since  $\langle 0 | a_i a_i^\dagger | 0 \rangle = 1$ ,  $\forall i \in S$ .

Let us consider two-particle states  $(\alpha a_{i_1}^\dagger a_{i_2}^\dagger + \beta a_{i_2}^\dagger a_{i_1}^\dagger) | 0 \rangle$ ,  $\alpha, \beta \in \mathcal{C}$ ,  $i_1, i_2 \in S$ .

If  $i_1 \neq i_2$ , the norm is

$$|\alpha|^2 + |\beta|^2 + \alpha^* \beta q_{i_1 i_2} + \alpha \beta^* q_{i_2 i_1} \quad (14)$$

Hence the norms of all two-particle states will be positive if and only if the matrices  $A^{(i_1,i_2)}$  are positive definite for  $\forall i_1, i_2 \in S$ . These conditions are (see Eq(13))

$$1 - |q_{i_1 i_2}|^2 > 0, \quad i.e. \quad |q_{i_1 i_2}| < 1 \quad i_1 \neq i_2; \quad i_1, i_2 \in S. \quad (15)$$

When  $i_1 = i_2 = i$ , the norms are positive if



$$A^{(i,i)} = 1 + q_{ii} > 0, \text{ i.e., } q_{ii} > -1, \quad \forall i \in S. \quad (16)$$

Note that when the inequalities (15) are satisfied, the two particle states  $a_{i_1}^\dagger a_{i_2}^\dagger |0\rangle$ , and  $a_{i_2}^\dagger a_{i_1}^\dagger |0\rangle$  are linearly independent. The norms of symmetric and antisymmetric two-particle states  $\|\frac{1}{2}(a_{i_1}^\dagger a_{i_2}^\dagger \pm a_{i_2}^\dagger a_{i_1}^\dagger)|0\rangle\|^2$  are  $\frac{1}{2}(1 \pm \text{Re} q_{i_1 i_2})$ , where the upper (lower) sign corresponds to the symmetric (antisymmetric) state. However, these states are not orthogonal if  $\text{Im} q_{i_1 i_2} \neq 0$ . The eigenstates of  $A^{(i_1, i_2)}$  are  $\frac{1}{2}(a_{i_1}^\dagger a_{i_2}^\dagger \pm e^{-i\phi} a_{i_2}^\dagger a_{i_1}^\dagger)|0\rangle$ ,  $\phi = \text{Im} q_{i_1 i_2}$ , and they are orthogonal. The occupation probabilities that the state  $a_{i_1}^\dagger a_{i_2}^\dagger |0\rangle$  is in these eigenstates are  $\frac{1}{2}(1 \pm |q_{i_1 i_2}|)$ , respectively.

Proceeding in this way we demand that every n-particle state should have positive norm. We find that the norms of all n-particle states will be positive if and only if the matrices  $A^{(i_1, \dots, i_n)}$  are positive definite for every ordered sequence of indices  $i_1, \dots, i_n$ ,  $i_1 \leq i_2 \leq \dots \leq i_n$ .

Furthermore, we point out that for  $q_{ij} = 0$ ,  $\forall i, j \in S$ , all conditions for positive definiteness are satisfied automatically, since  $A$  is the identity matrix. There arises a natural question how far one can change  $q_{ij}$ ,  $\forall i, j \in S$ , starting from zero in order that all eigenvalues remain positive, i.e., all matrices  $A^{(i_1, \dots, i_n)}$  remain positive definite. This will happen as long as  $\det A^{(i_1, \dots, i_n)} > 0$  for  $\forall i_1, \dots, i_n \in S$ ,  $i_1 \leq i_2 \leq \dots \leq i_n$  since the eigenvalues of  $A^{(i_1, \dots, i_n)}$  are real and depend continuously on  $q_{ij}$ ,  $\forall i, j \in S$ .

The determinant of a  $n! \cdot n!$  generic matrix  $A^{(i_1, \dots, i_n)}$ ,  $i_1 < i_2 < \dots < i_n$ , is given by [14]

$$\det A^{(i_1, \dots, i_n)} = \prod_{k=1}^{n-1} \left\{ \prod_{(j_1, \dots, j_{k+1})} [1 - \prod_{\alpha} \prod_{\beta} |q_{j_{\alpha} j_{\beta}}|^2] \right\}^{(k-1)!(n-k)!} \quad (17)$$

where  $1 \leq \alpha < \beta \leq k+1$ , and the second product is over

$$\binom{n}{k+1}$$

combinations of indices  $(j_1, \dots, j_{k+1}) \subset (i_1, \dots, i_n)$ . When  $q_{ij} = q \in \mathcal{R}$ ,  $\forall i, j \in S$ , Eq.(17) reproduces the result of Zagier [13]. We have not found a general expression for the determinant of the reduced matrix  $A^{(i_1, \dots, i_n)}$  when some of the indices coincide. However, the determinant of the reduced matrix  $A^{(i_1, \dots, i_n)}$  when some indices coincide is a polynomial factor of the determinant of the generic matrix, Eq.(17), after identifying the corresponding indices [14]. Hence, a sufficient condition for the positivity of all norms is that the expression in Eq.(17) should be positive for every ordered sequence  $i_1, \dots, i_n \in S$  including repetitions and for  $\forall n$ . It follows that the Fock-like space is positive definite for  $|q_{ij}| < 1$ ,  $\forall i, j \in S$ .

Note that in the three-particle sector the positivity of the reduced matrix  $A^{(iij)}$ , when two of the indices coincide, implies  $-1 < q_{ii} < |q_{ij}|^{-2}$ ,  $\forall j \in S$ ,  $j \neq i$ . Hence, in some cases, one might expect that  $q_{ii}$  could be larger than 1.

(ii) The number operator  $N_k$  exists for generalized quons, satisfying  $[N_k, a_l] = -a_k \delta_{kl}$ . We present a new result for the number operator [15] that exhibits its simple structure and is given by

$$N_k = a_k^\dagger a_k + \sum_{n=1}^{\infty} \sum_{i_1, \dots, i_n} \sum_{\pi \in S_n} [Y_{k, \pi(i_1, \dots, i_n)}]^\dagger Y_{k, i_1, \dots, i_n} (A^{(k, i_1, \dots, i_n)})_{k, i_1, \dots, i_n; k, \pi(i_1, \dots, i_n)}^{-1}, \quad (18)$$

where  $S_n$  denotes the permutation group, and

$$Y_{k, i_1, \dots, i_n} = Y_{k, i_1, \dots, i_{n-1}} a_{i_n} - q_{i_n k} q_{i_n i_1} \cdots q_{i_n i_{n-1}} a_{i_n} Y_{k, i_1, \dots, i_{n-1}} \quad (19)$$

and

$$Y_{k i_1} = a_k a_{i_1} - q_{i_1 k} a_{i_1} a_k. \quad (20)$$

Always when the parameters  $q_{ij}$  tend to 1, i.e.,  $q_{ij} \rightarrow 1$ , for  $\forall i, j$ , quons tend to a particular anyonic-type statistics, the matrices  $(A^{(i_1, \dots, i_n)})^{-1}$  become singular and the coefficients in the number operator, Eq.(18), diverge. Nevertheless, the number operator  $N_k$ , when acting on states, is well defined. Moreover, in the exact limit it reduces to  $N_k = a_k^\dagger a_k$ , and additional relations between the annihilation (creation) operators  $a_i, a_j$  ( $a_i^\dagger, a_j^\dagger$ ) emerge. In this case, the corresponding particles are not distinguishable, i.e., they are identical in the quantum-mechanical sense. Interchanging them, we generally obtain a unit phase factor  $e^{i\alpha}$  (typical of anyons). The n-particle wave functions  $a_{i_1}^\dagger \cdots a_{i_n}^\dagger |0\rangle$  for different permutations of indices are linearly dependent. Symmetries of these states can be classified according to the q-symmetrization or q-antisymmetrization prescription. Otherwise, for general quons there are  $n!$  linearly independent states  $a_{i_1}^\dagger \cdots a_{i_n}^\dagger |0\rangle$ , for different permutations of indices  $1, \dots, n$ .

(iii) The theory of generalized quons is nonlocal. The TCP theorem and the clus-

tering properties still hold, with the same type of arguments as in Ref.1 and 2. However, a careful relativistic treatment of generalized quons should be undertaken, see the discussion in Refs.16 and 17.

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